Volterra Series

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Abstract

The following are notes on what I've taught myself about Volterra seriesss. So it's probably all wrong.

1 Motivation

Well, I'm trying to model a power amplifier in a non-linear way. I learned that you can't model a non-linear component as a linear time-invariant system since it's not linear. Check it out:

You can represent any system like this:

$$y(n) = T[x(n)]. \tag{1}$$

Where $T[\cdot]$ is how the system acts on the input, $x(n)$.

For a system to be linear, superposition has to work, like this:

$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]. \tag{2}$$

aka the output of the system when you put two things in it is the same as when you put them both in seperately and add their outputs later on. Even kindergardeners know that in a non-linear PA, $x_1(n)$ and $x_2(n)$ interact to create things like harmonics. So, we can't use all those things everybody knows about LTI systems. So we're screwed.

2 Vito Volterra to the rescue!

If you can look at figure 1 and not start shaking out of fear for being non-linear and knowing what Vito's got for you, you're foolish.

Enough fooling around, time for the math.

Figure 1: Time to kick ass

3 First order Volterra series

This is RUF STUF, so we'll ease into it by learning first order systems first. A first order Volterra series is the same thing as a linear Volterra series. Actually, it's just a convolution. So it's not too shabby, but it's not that useful for what we need. Anyways, here goes:

$$y(t) = \int_{-\infty}^{+\infty} x(\sigma) h_1(t-\sigma)\, d\sigma. \quad (3)$$

Yeah, that's not so bad. Anyways, the important thing here is that this is pretty much the same as the convolution integral. That means that $h_1(t)$ is the impulse response of the system. And OBVIOUSLY $x(\sigma)$ is the input.

3.1 Example

This example is a modified version of chapter 2 in Schetzen.

LOOK OUT!! Here comes a RC circuit (Figure 2):

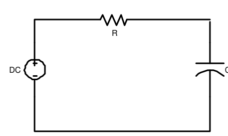

Figure 2: RC circuit

We all know the impulse response of this circuit:

$$h(t) = RCe^{-\frac{t}{RC}}. \quad (4)$$

And of course the output, $y(t)$ is the voltage across the capacitor. Oh, and we need an input signal, which we'll get to later. Look for something called $x(\sigma)$.

So lets do a real example of using a first orderVolterra series to model this system. First, we put our impulse response into equation 3:

$$y(t) = \int_{-\infty}^{+\infty} x(\sigma) RCe^{-\frac{t-\sigma}{RC}}\, d\sigma. \quad (5)$$

Things will look nicer if we say,

$$a = RC. \tag{6}$$

And rewrite that equation to look like this:

$$y(t) = \int_{-\infty}^{+\infty} x(\sigma) a e^{-\frac{(t-\sigma)}{a}}\, d\sigma. \tag{7}$$

Ok, now all we need is an input. Let's use this:

$$x(\sigma) = \begin{cases} 1, & \text{it } |\sigma| < T \\ 0, & \text{elsewhere} \end{cases} \tag{8}$$

Which of course looks like figure 3, when $T = 1$.

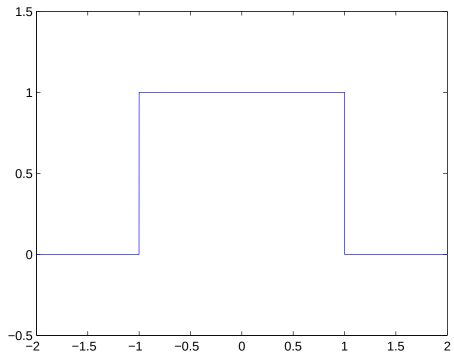

Figure 3: Put that in your pipe

Now, if you think about the guts of the integral in equation 7, you can see that it's only nonzero when $\sigma > -T$, so we don't have to worry about it at any other time, meaning that we can change the limits of the integral. We still ought to do it in two parts though, since it's different when $-T < t < T$ and when $t > T$. Here's the first one of those:

$$y(t) = \int_{-T}^{t} a e^{-\frac{1}{a}(t-\sigma)}\, d\sigma. \tag{9}$$

Ok, now split up that exponent and take out those constants to get:

$$y(t) = a e^{-\frac{t}{a}} \int_{-T}^{t} e^{\frac{1}{a}\sigma}\, d\sigma. \tag{10}$$

Which isn't a very hard integral, and solves to:

$$y(t) = 1 - e^{-\frac{1}{a}(t+T)}. \tag{11}$$

Remember that this is only when $-T < t < T$. For the other case, $t > T$ we do pretty much the same thing, except with different limits on the integral:

$$y(t) = \int_{-T}^{T} ae^{-\frac{1}{a}(t-\sigma)}\, d\sigma. \tag{12}$$

And we get:

$$y(t) = \left(e^{\frac{T}{a}} - e^{-\frac{T}{a}}\right) e^{-\frac{t}{a}}. \tag{13}$$

For $t > T$.

Now lets actually put numbers in and graph this junk. Let's see, how about $R = 1$, $C = 1$, and like we said before, $T = 1$. That makes everything easy since all of the constants are then 1.

Take a look at figure 4

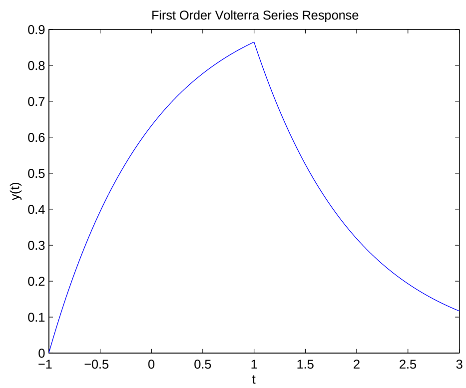

Figure 4: Get the lead out

If you think about a capacitor, this looks right. When the power gets turned on at $t = -1$, the capacitor starts charging up until the power gets turned off at $t = 1$ and it starts discharging. Somebody put a tent on that circus!

4 Second order system

Piranhas are a tricky species. So are second order Volterra systems. You sort of just do the same thing as a first order system twice. Take a gander:

$$y(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2)\, d\tau_1 d\tau_2. \tag{14}$$

See, it looks pretty much the same. Except that this $h_2(\tau_1, \tau_2)$ gives me the heebie-jeebies. What is that supposed to mean? 2-dimensional impulse response? Yeah, it sort of does. I know, it's creepy.

This is a pretty good place to remember that there's two forms of a convolution integral. I used one form earlier, and I'm using the other now. The first one is:

$$y(t) = \int h(\tau)x(t-\tau)\,d\tau, \tag{15}$$

and the other is:

$$y(t) = \int x(\sigma)h(t-\sigma)\,d\sigma. \tag{16}$$

And I guess they're the same thing, it's just that sometimes one can be easier to manipulate than the others. It makes intuitive sense, since to do a convolution you slide the input over the impulse response, so why can't you slide the impulse response over the input?

4.1 Example with no numbers

Here's an example to calm us down. Take a look at figure 5. That's a system that has linear and non-linear components. the linear part is the $h(t)$ and the non-linear part is the squarer.

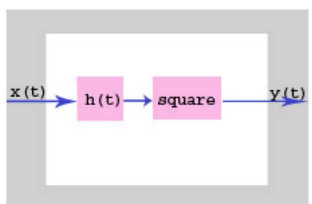

Figure 5: Square it up

So what's the Volterra series representation for this mess? Let's figure it out:

Ok, this equation we get from looking at the system and seeing that the output, $y(t)$ is the square of the output of the linear term which we can get from the section on first order systems. So then the output of this system is just equation 4 squared.

$$y(t) = \left(\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau\right)^2. \tag{17}$$

Sweet.

Let's keep going with this and multiply that out, using dummy variables to keep everything separated.

$$y(t) = \int_{-\infty}^{+\infty} h(\tau_1)x(t-\tau_1)\,d\tau_1 \int_{-\infty}^{+\infty} h(\tau_2)x(t-\tau_2)\,d\tau_2. \tag{18}$$

That's not too shabby. Now lets jam those two integrals together:

$$y(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2)\, d\tau_1 d\tau_2. \tag{19}$$

Now is when the creepiness of this whole scene subsides. Compare this equation with equation 4. They mean the same thing, so we can see that

$$h_2(\tau_1, \tau_2) = h(\tau_1)h(\tau_2). \tag{20}$$

So for this case, that 2-dimensional impulse response that frightened us earlier turns out to just be the regular impulse response multiplied by itself with a different variable in it's guts. See, he's not so bad, he's a pussycat. So if you know the linear impulse response and the input, you can get the output. We'll do that with numbers in the next section.

4.2 Example with numbers

Let's try and do something useful today. We'll find the Volterra series response for a circuit with impulse response,

$$h(t) = RCe^{-\frac{t}{RC}}u(t). \tag{21}$$

That should look familiar since it's the same one we did earlier. But wait! Let's get loco and do a second order Volterra response!?!

If we can remember as far back as the previous section, we see that we can use this impulse response to get the second order impulse response for our Volterra series. Look at equation 20. We can just stick dummy variables in our $h(t)$ and do a little multiplication to get:

$$h_2(\tau_1, \tau_2) = \left[RCe^{-\frac{\tau_1}{RC}}\right]\left[RCe^{-\frac{\tau_2}{RC}}\right] \tag{22}$$

Note that we can use the same R and C since it's actually the same resistor and capacitor.

Let's call $RC = b$ and multiply those together so it looks simpler and so that I don't have to type so much,

$$h_2(\tau_1, \tau_2) = b\left(e^{-\frac{1}{b}(\tau_1-\tau_2)}\right). \tag{23}$$

That might be wrong since I'm bad at algebra. But let's assume it's right.

Now we can jam that result into equation 14 (or equation 19, it doesn't matter) like this,

$$y(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} be^{-\frac{1}{b}(\tau_1-\tau_2)}x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2. \tag{24}$$

I can't remember how to do integrals right now, so I'm going to let this wait and update it later. But all that's left to get an expression for $y(t)$ is to come up with an input (I'll probably do a stovepipe like in the previous example) and solve the integral.

5 Too Many H's

Schetzen uses the letter H for pretty much everything. I think his wife's name must be Helga. I'll write down the differences to keep them straight.

Functional representation of a second-order Volterra operator (Note square brackets and single parameter):

$$\mathbf{H}_2[x(t)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\sigma_1)x(\sigma_2)h_2(t-\sigma_1, t-\sigma_2)\, d\sigma_1 d\sigma_2. \tag{25}$$

Time-invariant bilinear operator functional representation:

$$\mathbf{H}_2\{x_1(t), x_2(t)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(\tau_1, \tau_2)x_1(t-\tau_1)x_2(t-\tau_2)\, d\tau_1 d\tau_2. \tag{26}$$

As always, $\tau = t - \sigma$, so the previous equation can be written like:

$$\mathbf{H}_2\{x_1(t), x_2(t)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1(\sigma_1)x_2(\sigma_2)h_2(t-\sigma_1, t-\sigma_2)\, d\sigma_1 d\sigma_2. \tag{27}$$

So it looks like the whole thing for the second order operator is:

$$y(t) = \mathbf{H}_2[x_1(t)] + \mathbf{H}_2[x_2(t)] + 2\mathbf{H}_2\{x_1(t), x_2(t)\}. \tag{28}$$

You can see where this whole thing becomes a real mess. Anything past a 3rd order system is pretty much intractable. For a second order system, you've got to figure out all 3 parts of equation 25 and add them together. Looking at the definitions of all those parts, you're looking down the barrel of 3 double integrals. To be useful for nonlinear modeling, you'd usually want to use like a 7th order system, and I'm not even going to think about any septuple integrals.